Chaos-nonchaos phase transitions induced by external noise in ensembles of nonlinearly coupled oscillators

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Nonlinear dynamical behaviors of ensembles of nonlinearly coupled oscillators subjected to external noise are studied on the basis of nonlinear Fokker-Planck equations. The effects of two kinds of noise, the Langevin noise and the noise introduced in the coupling strength, are investigated, and phase transitions involving chaos-nonchaos bifurcations are found to occur as the noise level is changed. An H theorem is proposed for the nonlinear Fokker-Planck equation to ensure stability of the Gaussian type solution that is approached for large times.

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I. INTRODUCTION

The behaviors of nonlinear dynamical systems subjected to external noise has attracted much attention from researchers in many areas of science [1,2]. While the effects of noise are usually considered to obscure the fine structure of intrinsic physical properties of a system, or to deteriorate the degree of coherence or order, the phenomena of stochastic resonance [2,3] are known to reveal a favorable aspect of the action of noise in improving the performance of the system, as is shown by the fact that noise can enhance the response of a bistable system to a weak time-periodic signal under certain conditions. The degree of stochastic resonance may be made pronounced by introducing appropriate couplings among nonlinear elements to make a coupled system [4,5] (see also [6]), whose behaviors have also been of great interest in recent years from the viewpoint of developing a neural network theory [7–12].

Unlike the case of a system of a single or a small number of element(s) or oscillator(s), a coupled system composed of a great number of elements becomes robust to a certain extent against the influence of noise with respect to the preservation of the intrinsic natures of the constituents [13,14]: whereas a damped oscillator with a symmetric bistable potential that is subjected to external white noise exhibits ergodicity without showing any bifurcations, a system of N coupled damped oscillators with mean-field couplings can undergo a ferromagnetic-paramagnetic type phase transition in the thermodynamic limit $N\rightarrow\infty$, as the noise level is varied [15–17], giving rise to the occurrence of spontaneous symmetry breaking.

The occurrence of such a phase transition was studied on the basis of the analysis of a nonlinear Fokker-Planck equation (NFPE) [15–18] that describes the empirical probability distribution for a system of globally coupled Langevin equations under the thermodynamic limit. The NFPE emerges as a direct consequence of applying the law of large numbers in accordance with the scheme of mean-field coupling. Unlike the linear Fokker-Planck equation, the NFPE, which is a kind of nonlinear master equation, is no longer expected to exhibit ergodicity and hence may give rise to the occurrence of bifurcations of solutions [5,13–18]. The NFPE with a symmetric bistable potential and a diffusive coupling taken into account indeed exhibits a pitchfork bifurcation representing a ferromagnetic-paramagnetic type phase transition, while an H theorem has been shown to still hold to ensure global stability of the system [17]. In this case the coupled system satisfies the so-called detailed balance condition leading to the existence of an energy function.

The problem of studying the case without the detailed balance condition then arises, to see what will happen to such a system as the noise level is changed. The effects of noise on synchronization phenomena in coupled limit cycle oscillators [13,14] as well as in oscillator neural networks [11,12] have been studied, using the NFPE, to confirm that order-disorder type phase transitions can occur. It will be of particular interest to observe behaviors of a system of coupled chaotic oscillators, since the appearance or disappearance of chaos with changes of noise level may be viewed as a way of controlling chaos with noise [19,20]. Recently, the effects of noise on a coupled map lattice with mean-field couplings have been studied by Shibata, Chawanya, and Kaneko [21] using computer simulations on the nonlinear Perron-Frobenius equation [22]. They have shown that noise reduces the degree of complexity in terms of the Lyapunov dimension of the system. The result that too much noise added to the system leads to the creation of a fixed point in the space of order parameters is consistent with the appearance of the disordered phase observed in the order-disorder type phase transition of coupled limit cycle oscillator systems [12]. Dealing with the case of a coupled chaotic system of time-continuous oscillators will, however, require conducting extensive numerical work based on time-consuming computer simulations.

The aim of this paper is to study the effect of noise on globally coupled systems exhibiting time-continuous chaotic oscillations as rigorously as possible on the basis of analytically solvable models. To this end we consider nonlinearly coupled oscillator systems taking the form of analog neural network equations that exhibit chaotic or limit cycle oscillations in the absence of external noise [23]. We introduce noise into such systems in two ways: by applying the Langevin noise as an external force in the standard way, and by adding colored noise in the coupling strength. Studies of the effects of noise introduced in the coupling strength from the viewpoint of phase transitions have been quite few. We investigate stochastic behaviors of the system by employing the method of the nonlinear Fokker-Planck equation. The NFPE corresponding to such systems yields a solution representing oscillatory motions of the probability distribution whose dynamical behavior may depend on the magnitude of the diffusion constants. We will show that an H theorem also holds for the NFPE of the present system to ensure stability of the Gaussian distribution that is given as its special solution.

II. MODEL AND NONLINEAR FOKKER-PLANCK EQUATION APPROACH

We begin by defining a system of N elements coupled via nonlinear global interactions and subjected to external noise, whose dynamics is described by a set of Langevin equations:

$$\dot{x}^{(i)} = -b_1 x^{(i)} + \sum_{j=1}^{N} J_1^{(ij)} V_1(a_{11}x^{(j)} + a_{12}y^{(j)} + a_{13}z^{(j)}) + f_1^{(i)}(t),$$

$$\dot{y}^{(i)} = -b_2 y^{(i)} + \sum_{j=1}^{N} J_2^{(ij)} V_2(a_{21}x^{(j)} + a_{22}y^{(j)} + a_{23}z^{(j)}) + f_2^{(i)}(t), \tag{1}$$

$$\dot{z}^{(i)} = -b_3 z^{(i)} + \sum_{j=1}^{N} J_3^{(ij)} V_3(a_{31}x^{(j)} + a_{32}y^{(j)} + a_{33}z^{(j)}) + f_3^{(i)}(t), \quad i=1,\ldots,N,$$

with $\langle f_k^{(i)}(t) f_l^{(j)}(t') \rangle = 2D_k \delta(t-t')\,\delta_{ij}\delta_{kl}$ $(D_k>0)$ and $b_1, b_2, b_3 > 0$, where a_{kl} and b_k are constants and V_k denote appropriate nonlinear functions specifying the nonlinear couplings. We assume V_k to be bounded functions such as $\tanh \beta(\cdot)$, $\sin(\cdot)$, and so on. The mean-field coupling strengths J_{ij} may be assumed to be given by

$$J_k^{(ij)} = \frac{1}{N}(\varepsilon_k + \varepsilon_k^{(i)}), \quad k=1,\ldots,3, \tag{2}$$

where ε_k are constants, and $\varepsilon_k^{(i)}$ represents appropriately defined colored noise in the coupling strength. For simplicity we take $\varepsilon_1^{(i)} = \varepsilon_2^{(i)} = 0$, and assume $\varepsilon_3^{(i)}$ to obey the Ornstein-Uhlenbeck process:

$$\frac{d}{dt}\varepsilon_3^{(i)} = -\gamma \varepsilon_3^{(i)} + f_\varepsilon^{(i)}(t),$$
$$\langle f_\varepsilon^{(i)}(t) f_\varepsilon^{(j)}(t') \rangle = 2D_4 \delta(t-t') \quad (\gamma>0,\ D_4 \geqslant 0). \tag{3}$$

Choosing such a sigmoidal function as $\tanh \beta(\cdot)$ with analog gain β for $V(\cdot)$, we can view Eq. (1) as a network model equation of analog neurons, which have been extensively studied in the case without noise [8–10]. In the absence of noise, an appropriate parameter setting for a_{kl} in Eq. (1) yields limit cycle or chaotic oscillations even for the case with finite N [23], for which the introduction of any small amount of external noise ($D_k>0$, $k=1,2,3$) would, however, bring ergodicity into the stochastic system to prevent oscillatory motions of averaged physical quantities. We are concerned with investigating the behaviors of the system driven by external noise in the thermodynamic limit $N\to\infty$. To this end it will be appropriate to transform the above set of equation into the nonlinear Fokker-Planck equation describing the time evolution of the empirical probability distribution:

$$\begin{aligned}\frac{\partial p(t,x,y,z,\varepsilon)}{\partial t} &= -\frac{\partial}{\partial x}[(-b_1 x + \varepsilon_1 \langle V_1 \rangle)p] \\ &\quad -\frac{\partial}{\partial y}[(-b_2 y + \varepsilon_2 \langle V_2 \rangle)p] \\ &\quad -\frac{\partial}{\partial z}[(-b_3 z + (\varepsilon_3 + \varepsilon)\langle V_3 \rangle)p] \\ &\quad -\frac{\partial}{\partial \varepsilon}[-\gamma \varepsilon p] + \left(D_1 \frac{\partial^2}{\partial x^2} + D_2 \frac{\partial^2}{\partial y^2} \right. \\ &\quad \left. + D_3 \frac{\partial^2}{\partial z^2} + D_4 \frac{\partial^2}{\partial \varepsilon^2}\right)p \\ &\equiv L_F[p(t)]p \end{aligned} \tag{4}$$

with

$$\langle V_k \rangle = \int V(a_{k1}x + \alpha_{k2}y + a_{k3}z) p(t,x,y,z,\varepsilon)\, dx\, dy\, dz\, d\varepsilon. \tag{5}$$

We then straightforwardly obtain the order parameter equation of the means $\langle x \rangle$, etc.:

$$\frac{d}{dt}\langle x \rangle = -b_1 \langle x \rangle + \varepsilon_1 \langle V_1 \rangle, \tag{6a}$$

$$\frac{d}{dt}\langle y \rangle = -b_2 \langle y \rangle + \varepsilon_2 \langle V_2 \rangle, \tag{6b}$$

$$\frac{d}{dt}\langle z \rangle = -b_3 \langle z \rangle + (\varepsilon_3 + \langle \varepsilon \rangle) \langle V_3 \rangle, \tag{6c}$$

$$\frac{d}{dt}\langle \varepsilon \rangle = -\gamma \langle \varepsilon \rangle. \tag{6d}$$

Note that the NFPE has the characteristic feature that one can formally separate it into the motions of the means $\langle x \rangle$, $\langle y \rangle$, $\langle z \rangle$, and $\langle \varepsilon \rangle$ and the second moments of the probability distribution around the mean, although the former is in reality affected by the latter via the term $\langle V_k \rangle$ that arises from the couplings. Indeed, putting $u = x - \langle x \rangle$, $v = y - \langle y \rangle$, $w = z - \langle z \rangle$, $\eta = \varepsilon - \langle \varepsilon \rangle$, one obtains the moment equations

$$\frac{d}{dt}\langle w^2 \rangle = -2b_3 \langle w^2 \rangle + 2\langle w\eta \rangle \langle V_3 \rangle + 2D_3, \tag{7a}$$

$$\frac{d}{dt}\langle w\eta\rangle = -(b_3+\gamma)\langle w\eta\rangle + \langle \eta^2\rangle\langle V_3\rangle, \tag{7b}$$

together with

$$\frac{d}{dt}\langle u^2\rangle = -2b_1\langle u^2\rangle + 2D_1, \quad \frac{d}{dt}\langle v^2\rangle = -2b_2(v^2) + 2D_2,$$

$$\frac{d}{dt}\langle u\eta\rangle = -(b_1+\gamma)\langle u\eta\rangle, \quad \frac{d}{dt}\langle v\eta\rangle = -(b_2+\gamma)\langle v\eta\rangle,$$

$$\frac{d}{dt}\langle uv\rangle = -(b_1+b_2)\langle uv\rangle,$$

$$\frac{d}{dt}\langle vw\rangle = -(b_2+b_3)\langle vw\rangle + \langle v\eta\rangle\langle V_3\rangle, \tag{8}$$

$$\frac{d}{dt}\langle wu\rangle = -(b_1+b_3)\langle wu\rangle + \langle u\eta\rangle\langle V_3\rangle,$$

$$\frac{d}{dt}\langle \eta^2\rangle = -2\gamma\langle \eta^2\rangle + 2D_4,$$

where $\langle\cdot\rangle$ represents the average with respect to the corresponding probability distribution $\tilde{p}(t,u,v,w,\eta)$. Equation (8) takes a closed form without the need to know higher moments, since $\tilde{p}(t,u,v,w,\eta)$ turns out to obey a linear Fokker-Planck equation, given that $\langle V_k\rangle$ is known as a function of *t*. It then follows that a special solution to the linear Fokker-Planck equation can be given by a Gaussian distribution of the form

$$\tilde{p}_G(t,u,v,w,\eta) = \frac{1}{(2\pi)^2\sqrt{\det C(t)}}\exp\{-\tfrac{1}{2}(C(t)^{-1}\vec{s},\vec{s})\}, \tag{9}$$

where $\vec{s}=(u,v,w,\eta)$ and the matrix $C(t)$ has the components $C(t)_{ij}=\langle s_i s_j\rangle$ $(i,j=1,\ldots,4)$ that are the solution to the moment equations (7) and (8) with s_i denoting the component of $\vec{s}$.

When $\tilde{p}(t,u,v,w,\eta)=\tilde{p}_G(t,u,v,w,\eta)$ holds, by defining $m_k=a_{k1}x+a_{k2}y+a_{k3}z$, one can rewrite $\langle V_k\rangle$ in Eq. (5) as

$$\langle V_k\rangle = \int_{-\infty}^{\infty} V_k(\langle m_k\rangle + \tilde{m}_k)\frac{1}{\sqrt{2\pi}\sigma_k}\exp\left[-\frac{\tilde{m}_k{}^2}{2\sigma_k{}^2}\right]d\tilde{m}_k, \tag{10}$$

where

$$\sigma_k^2 = \langle (m_k - \langle m_k\rangle)^2\rangle,$$

$$\langle m_k\rangle = a_{k1}\langle x\rangle + a_{k2}\langle y\rangle + a_{k3}\langle z\rangle.$$

For large times it follows from Eq. (8) that

$$\langle \varepsilon\rangle = 0, \quad \langle\eta\rangle^2 = \frac{D_4}{\gamma}, \quad \langle u^2\rangle = \frac{D_1}{b_1}, \quad \langle v^2\rangle = \frac{D_2}{b_2}, \tag{11}$$

$$\langle u\eta\rangle = 0,$$

$$\langle v\eta\rangle = 0, \quad \langle uv\rangle = 0, \quad \langle vw\rangle = 0, \quad \langle wu\rangle = 0.$$

We see that Eqs. (6a)–(6c) and (7) with $\langle\varepsilon\rangle=0$ and $\langle\eta^2\rangle = D_4/\gamma$ exhaustively describe the dynamical behaviors of the coupled system for large times, for which

$$\sigma_k{}^2 = a_{k1}{}^2\frac{D_1}{b_1} + a_{k2}{}^2\frac{D_2}{b_2} + a_{k3}{}^2\langle w^2\rangle. \tag{12}$$

The Gaussian distribution $\tilde{p}_G(t,u,v,w,\eta)$ assumed in the above argument turns out to be approached with time elapsed from any initial condition for $p(t,x,y,z,\varepsilon)$. This can be more directly confirmed by the following *H* theorem for the NFPE.

H theorem. Let $p(t,x,y,z,\varepsilon)$ be any solution to the NFPE (4). Suppose that the Gaussian distribution $p_G(t,x,y,z,\varepsilon) \equiv \tilde{p}_G(t,x-\langle x\rangle,y-\langle y\rangle,z-\langle z\rangle,\varepsilon-\langle\varepsilon\rangle)$ in Eq. (9) is defined such that the means $\langle x\rangle$, etc., and covariances $\langle u^2\rangle$, etc., are given as an arbitrary solution to the moment equations (6), (7), and (8) with the average $\langle V_k\rangle$ taken with respect to the $p(t,x,y,z,\varepsilon)$ [Eq. (5)]. Then the following inequalities hold:

$$H(p,p_G) \equiv \int_{R^4} p\ln\left(\frac{p}{p_G}\right)dx\,dy\,dz\,d\varepsilon \geqslant 0, \tag{13}$$

$$\frac{d}{dt}H(p,p_G) \leqslant 0. \tag{14}$$

Proof. The first inequality follows from the entropy inequality $x-1\geqslant \ln x$ with $x=p_G/p$. To prove the second inequality, note that $p(t,x,y,z,\varepsilon)$ and $p_G(t,x,y,z,\varepsilon)$ satisfy the time evolution equation with the same Fokker-Planck operator $L_F[p(t)]$: $\partial p/\partial t = L_F[p(t)]p$, $\partial p_G/\partial t = L_F[p(t)]p_G$. Performing integration by parts one obtains

$$\frac{d}{dt}H(p,p_G) = -\int_{R^4} p\sum_{i=1}^{4} D_i\left\{\frac{\partial}{\partial x_i}\ln\left(\frac{p}{p_G}\right)\right\}^2\prod_i dx_i \leqslant 0, \tag{15}$$

where the notation $(x_1,x_2,x_3,x_4)=(x,y,z,\varepsilon)$ has been introduced.

The above *H* theorem implies that $p(t,x,y,z,\varepsilon)$ approaches $p_G(t,x,y,z,\varepsilon)$ for sufficiently large *t*, while the relative entropy monotonically decreases. Then it follows that the Gaussian distribution $p_G(t,x,y,z,\varepsilon)$ itself can be a solution of the NFPE under the certain conditions: $\partial p_G/\partial t = L_F[p_G(t)]p_G$ instead of $\partial p_G/\partial t = L_F[p(t)]p_G$. This is indeed the case, when the initial distribution $p(0,x,y,z,\varepsilon)$ is Gaussian: suppose one has $p(0,x,y,z,\varepsilon)=\delta(x-x_0, y-y_0, z-z_0)\sqrt{\gamma/2\pi D_4}\exp(-\gamma\varepsilon^2/2D_4)$ with the colored noise $\varepsilon_3^{(i)}$ being stationary.

Note that since $p_G(t,x,y,z,\varepsilon)$ depends on $p(t,x,y,z,\varepsilon)$, its uniqueness is not ensured, unlike the case of the usual *H* theorem for linear Fokker-Planck dynamics. Instead, one may expect bifurcation phenomena to occur as parameters involved in Eq. (4) are varied. We proceed to examine the occurrence of such phenomena with changes in the noise strength D_k $(k=1,\ldots,4)$.

The effect of the external noise D_k $(k=1,2,3)$ manifests itself through $\langle V_k\rangle$, which is determined by the variance σ_k^2.

The effective nonlinear coupling function $\langle V_k \rangle$, in general, becomes more smoothly shaped than the bare one V_k, as a result of the transformation (10).

When V_k is a sigmoidal function, $V_k(x)=1$ $(x>\theta_k)$, -1 $(x<-\theta_k)$, x/θ_k $(|x|<\theta_k)$, $\langle V_k \rangle$ takes the form of the smoother sigmoidal function

$$\langle V_k \rangle = \left(1+\frac{\langle m_k \rangle}{\theta_k}\right) N\left[\frac{\theta_k+\langle m_k \rangle}{\sigma_k}\right] + \left(-1+\frac{\langle m_k \rangle}{\theta_k}\right) N\left[\frac{\theta_k-\langle m_k \rangle}{\sigma_k}\right] + \frac{\sigma_k}{\sqrt{2\pi}\theta_k}\left[\exp\left(-\frac{(\theta_k+\langle m_k \rangle)^2}{2\sigma_k{}^2}\right) - \exp\left(-\frac{(\theta_k-\langle m_k \rangle)^2}{2\sigma_k{}^2}\right)\right], \tag{16}$$

where $N[x]=\int_0^x (1/\sqrt{2\pi})\exp(-z^2/2)dz$.

In the case with $V_k(x)=\sin x$, one can explicitly write down the average $\langle V_k \rangle$ as

$$\langle V_k \rangle = \exp\left(-\frac{\sigma_k{}^2}{2}\right)\sin\langle m_k \rangle, \tag{17}$$

which implies that the effect of noise can be viewed as simply reducing the original coupling strength ε_k by the factor $\exp(-\sigma_k^2/2)$.

III. NONEQUILIBRIUM PHASE TRANSITIONS AND CONTROL OF CHAOS BY NOISE

For simplicity we choose $V_k(x)=\sin x$ in this paper. Solving Eqs. (6a)–(6c) and (7) with $\langle \varepsilon \rangle = 0$ and $\langle \eta^2 \rangle = D_4/\gamma$, together with Eqs. (12) and (17), we can find the occurrence of various types of bifurcation. We assume $D_1=D_2=D_3$ $(=D)$ and $b_1=b_2=1$ for simplicity in what follows. Note that $\langle x \rangle = \langle y \rangle = \langle z \rangle = 0$ is always a solution, which is stable for D larger than a certain critical value D_c. Such a fixed point attractor corresponds to a paramagnetic phase with the highest symmetry in the ferromagnetic-paramagnetic type of thermodynamic phase transition [13–17]. For $D<D_c$, in addition to nonzero fixed point solutions, dynamical attractors such as the limit cycle and chaos may appear as a result of a stability switch when D is varied. It is noted that, since the system has inversion symmetry with respect to the transformation $(\langle x \rangle,\langle y \rangle,\langle z \rangle,\langle w^2 \rangle,\langle w\eta \rangle) \to (-\langle x \rangle,-\langle y \rangle,-\langle z \rangle,\langle w^2 \rangle,-\langle w\eta \rangle)$, two solutions symmetric to each other, if different, turn out to coexist.

First we set $D_4=0$ to observe the effect of changing D alone. In this case it will suffice to deal only with Eqs. (6a)–(6c) with $\langle \varepsilon \rangle = 0$ and

$$\langle V_k \rangle = \exp\left[-\frac{D}{2}\left(a_{k1}{}^2+a_{k2}{}^2+\frac{a_{k3}{}^2}{b_3}\right)\right]\sin\langle m_k \rangle,$$

since $\langle w^2 \rangle = D_3/b_3$ for large times. We show a typical example in Fig. 1, where the largest Lyapunov exponent λ_M is plotted against D in the case when the coupled system (1) with $D=0$ exhibits chaotic oscillations. We see that as D is increased from $D=0$ chaotic oscillations repeatedly undergo transitions from chaos to the limit cycle and then to chaos, including narrow periodic windows for entering a comparatively large D region of the limit cycle phase, which is followed by fixed point type attractor phases including the paramagnetic phase. As is expected from the above behavior of the bifurcations, if the system is chosen so as to exhibit limit cycle oscillations in the absence of noise, increasing D can bring about chaos in the system.

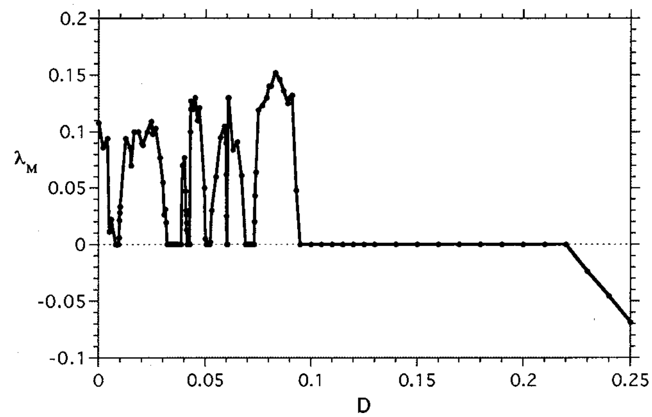

FIG. 1. Plot of the largest Liapunov exponents λ_M vs the noise strength D showing the occurrence of phase transitions in the stochastic systems (1) with $V_k(x)=\sin x$; the lines are guides to the eye. A positive value of λ_M implies that the motion is chaotic. Equations (6a)–(6c) were numerically integrated in the case where $a_{11}=1.0$, $a_{12}=-1.0$, $a_{13}=0.1$, $a_{21}=1.0$, $a_{22}=0.5$, $a_{23}=0.1$, $a_{31}=-3.0$, $a_{32}=0.6$, $a_{33}=0.93$, $D_4=0$, $\varepsilon_1=\varepsilon_2=1.55$, $\varepsilon_3=3\varepsilon_1$, $b_1=b_2=1$, and $b_3=1$. The ferromagnetic-paramagnetic transition point D_c (not shown here) is $D_c \approx 0.337$.

The appearance of chaos or limit cycle oscillations means that the motion of the average of the Gaussian distribution exhibits such oscillations, while the width of the distribution remains constant. However, when we consider the case with coupling noise, the appearance of oscillatory behavior of the system will be accompanied by oscillatory motions of both the location of the peak (average) and the width of the Gaussian distribution. Setting $D=0$, we investigate the D_4 dependence of the system. Figure 2 depicts the variation of the largest Lyapunov exponent λ_M with change in D_4 in the case where limit cycle oscillations appear in the absence of noise. We see the occurrence of bifurcations through periodic windows as D_4 is increased, from limit cycle to chaos and back to limit cycle attractors and finally to a chaotic phase, which remains in existence no matter how large the value of D_4. Indeed, such chaotic attractors for large D_4 are found to exhibit the D_4 dependence

$$(\langle x \rangle,\langle y \rangle,\langle z \rangle,\langle w^2 \rangle,\langle w\eta \rangle) = \left(O\left(\frac{1}{\sqrt{D_4}}\right),O\left(\frac{1}{\sqrt{D_4}}\right),O\left(\frac{1}{\sqrt{D_4}}\right),O(1),O(\sqrt{D_4})\right)$$

in the limit $D_4 \to \infty$ with the largest Lyapunov exponent tending to $\lambda_M \approx 0.024$.

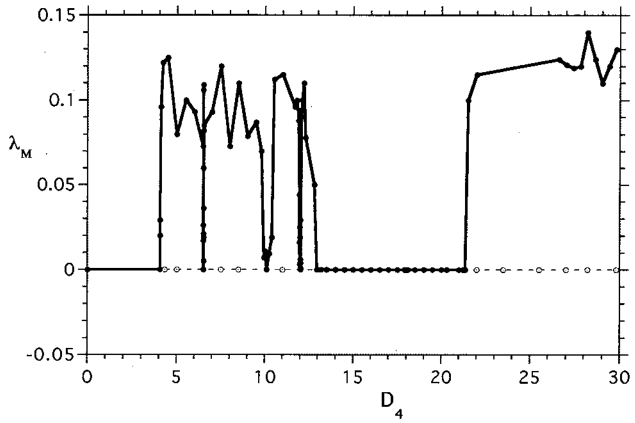

FIG. 2. Plot of the largest Liapunov exponents λ_M vs the noise strength D_4 displaying the effect of changing the noise level in the coupling strength, which is observed to be capable of inducing phase transitions. Equations (6a)–(6c), (7a), and (7b) with the same a_{ij} $(i,j=1,\ldots,3)$ were numerically integrated in the case where $D=0$, $\varepsilon_1=\varepsilon_2=1.7$, $\varepsilon_3=3\varepsilon_1$, $b_1=b_2=1$, $b_3=2$, and $\gamma=0.2$. Two different attractors (closed and open circles) coexist.

We also find that just after the onset of chaos at $D_4 \approx 4.06$ via a period doubling sequence a limit cycle circulating outside the chaotic trajectory manifests itself anew at $D_4 \approx 4.33$ to keep coexisting all the way for increasing D_4 with the chaos or limit cycle attractor lying inside it. It is worth noting that, unlike in the case of Fig. 1 with changing D, any further increase of D_4 does not induce a transition to the paramagnetic state with $\langle x\rangle=\langle y\rangle=\langle z\rangle=0$. This is because the presence of the coupling noise does not alter the linear stability of the solution $(\langle x\rangle,\langle y\rangle,\langle z\rangle,\langle w^2\rangle,\langle w\eta\rangle)=(0,0,0,D_3/b_3,0)$, as can easily be shown by a linear stability analysis of Eqs. (6a)–(6c) and (7), with the paramagnetic state remaining unstable for any value of D_4 in the case of assumed values of the parameters (Fig. 2).

IV. CONCLUDING REMARKS

We have shown the occurrence of nonequilibrium phase transitions involving the appearance or disappearance of chaotic attractors in nonlinearly coupled oscillatory systems subjected to external noise. Such phase transitions or bifurcations in stochastic systems as found in the present work can occur, in a strict sense, only in the thermodynamic limit. Noise added in the coupling strength has been found to bring about unusual behavior such that, even if the noise level is infinitely large, such nontrivial attractors as chaos or limit cycles may remain in existence.

One might be tempted to consider that, while a transition from chaos to limit cycle with increasing noise level, which can be viewed as the reduction of the degree of complexity due to noise, is quite reasonable, a reverse transition from limit cycle to chaos seems at first glance contradictory to the expected reduction of the degree of complexity. The transition from limit cycle to chaos induced by noise in our models is reminiscent of the phenomenon of noise-induced chaos that was previously reported, using digital and analog simulations, in the single Lorenz oscillator model. However, the former, which occurs only in the thermodynamic limit, will differ from the latter in character. Our present results reveal that problems of phase transitions of our model system can be simply as well as nicely understood in terms of legitimate bifurcations of the nonlinear dynamics with a few degrees of freedom reduced from the dynamics with infinitely many degrees of freedom. The analytical tractability of our model may be attributed to the specific feature that nonlinearity manifests itself only in the coupling terms of the original dynamical equations and that it becomes possible to conduct linear stochastic analyses of the Markov process in the thermodynamic limit.

We have also found that an H theorem with an appropriately defined H functional that takes the form of the relative entropy still holds for the NFPE capable of exhibiting bifurcations, and that it ensures stability of and convergence to the Gaussian distribution (9) that is given as a special solution to the NFPE. The H functional, which can be rewritten as

$$H(p,p_G)=\int_{R^4} p\ln p\,dx\,dy\,dz\,d\varepsilon+\tfrac{1}{2}\ln[\det C(t)] + 2\ln 2\pi+2, \tag{18}$$

however, differs from the usually known one, which takes the form of a free energy or of its increment measured with respect to the equilibrium state [17,24,25]. We have here assumed, in rewriting the above equation, that $\langle s_i s_j\rangle_p=\langle s_i s_j\rangle_{P_G}\equiv C(t)_{ij}\,(i,j=1,\ldots,4)$. The nonstandard form of Eq. (18) arises from the absence of the so-called detailed balance condition in the present system.

Details of the analysis including the behavior of the system in the large D_4 limit will be reported elsewhere.

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